A SIMPLE PHENOMENOLOGICAL HYPOTHESIS OF CUMULATIVE FATIGUE DAMAGE PART I. HALF-CYCLE FORMULATION

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A simple hypothesis of cumulative fatigue damage taking into account loading sequence effects is proposed for engineering purposes. Strictly phenomenological approach introducing notions of immediate and history damage factors is used to make the hypothesis simple, synthetic and having transparent and well analysable final mathematical expression. The hypothesis satisfactorily approximates constant amplitude fatigue lives and fatigue lives in H-L (high-low) and L-H (low-high) loading cases.

Key words: cumulative fatigue damage, hypothesis, loading sequence

JEDNODUCHÁ FENOMENOLOGICKÁ HYPOTÉZA KUMULATÍVNEHO ÚNAVOVÉHO POŠKODENIA ČASŤ I. FORMULÁCIA NA ZÁKLADE POLCYKLOV

Na inžinierske účely sa navrhuje jednoduchá hypotéza kumulácie únavového poškodenia zohľadňujúca účinky poradia zaťažovania. Použitím vyhranene fenomenologického prístupu, zavádzajúceho pojmy faktor okamžitého poškodenia a faktor histórie poškodenia, sa dosiahla jednoduchosť a syntetickosť hypotézy a jej transparentné a dobre analyzovateľné výsledné matematické vyjadrenie. Hypotéza uspokojivo aproximuje únavové životnosti pri zaťažovaní s konštantnou amplitúdou a únavové životnosti v prípade zaťažovania typu H-L a L-H.

1. Introduction

The oldest fatigue damage hypothesis is the Palmgren-Miner's one [1], known also as linear hypothesis. Fatigue lives estimated according to linear hypothesis are not always in agreement with experimental values. Therefore, many other rules based on different ideas of cumulative fatigue damage were suggested. Their

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summary can be found in the works [2–4] and especially in [5]. Fatigue damage hypotheses may be divided into two main groups: hypotheses based on physical fatigue damage ideas and hypotheses based on phenomenological fatigue damage ideas. Phenomenological hypotheses try to describe the fatigue process on the basis of its manifestations in fatigue lives at different types of loading. In engineering praxis, however, despite the fact that the rules of both groups often provide fatigue-life estimations that are more accurate and less non-conservative, the Palmgren--Miner hypothesis is still the most or often the only used one. There are several reasons for it. One of them is its simple and synthetic basic idea and consequently the simple interpretation of computed fatigue lives.

The aim of this work is to formulate the hypothesis of cumulative fatigue damage for engineering purposes, taking into account loading sequence effects, with particular emphasis on its basic idea to be simple, synthetic and resulting in a simple, transparent and well analysable final mathematical expression. For the time being, the phenomenological approach meets these requirements better especially in the fatigue-crack initiation and early growth stages, and so, we have chosen strictly phenomenological approach. It is obvious that phenomenological approach can not arrive to fully adequate description of fatigue damaging. Despite this, phenomenological approach provides not only important results from the practical point of view, but it may also offer inspiring ideas for investigation of the physical nature of the fatigue process.

2. Basic idea of cumulative fatigue damage hypothesis

The basic idea resides in assumption, according to which the increment of fatigue damage is influenced also by another quantity depending on previous loading, besides the effect of immediately affecting loading half-cycle. More specifically, it is supposed that the increment of fatigue damage D due to both quantities can be expressed in the form

$$\mathrm{d}D = H I_{\mathrm{hc}} \mathrm{d}(2N), \tag{1}$$

where I_{hc} is the immediate damage factor of half-cycle, H is the history damage factor and N is the number of loading cycles. The immediate damage factor of a half-cycle I_{hc} represents the immediate effect of just running ($2N$-th) loading half-cycle on fatigue damage, the history damage factor H is the effect of previous loading. It is obvious that I_{hc} is restricted by the condition, according to which I_{hc} must increase with increasing half-cycle range and, at the same time, if "half-cycle" range is zero, I_{hc} is equal to zero, as well. The history damage factor includes the effect of an individual half-cycle, e.g. effect of residual stress at crack tip after an overload, and also cumulative effect of several half-cycles in general.

It is supposed that the immediate damage factor can be expressed in the form

$$I_{\mathrm{hc}}(2N) = \Delta V^{\alpha}(2N), \tag{2}$$

where $\Delta V(2N)$ is a damage controlling quantity (e.g. plastic strain range), being the function of $2N$, and α is a constant. As for the damage controlling quantity, our considerations will be general. If the damage controlling quantity has a physical dimension, it is supposed to be given in a suitable relative form. The immediate damage factor satisfies restrictions mentioned when

$$\alpha > 0. \tag{3}$$

Further, it is supposed that the history damage factor can be expressed in the form

$$H(2N) = \int_{0}^{2N} \Delta V^{\beta}(2h)\mathrm{d}(2h), \tag{4}$$

where h denotes the number of cycles in the loading history and

$$\beta > 0 \tag{5}$$

is constant. H increases with increasing values of function $\Delta V(2h)$ and increasing number of cycles N. The history damage factor is thus a simple cumulative effect of all previous loading half-cycles.

Damage at fracture D_{f} is supposed to be constant, independent of magnitude and loading sequence,

$$D_{\mathrm{f}} = K, \tag{6}$$

where K is a constant. Substituting (2) and (4) into (1), integrating from beginning to the fracture and using Eq. (6), a general equation based on our hypothesis is obtained

$$\int_{0}^{2N_{\mathrm{f}}} \left(\int_{0}^{2N} \Delta V^{\beta}(2h)\mathrm{d}(2h) \right) \Delta V^{\alpha}(2N)\mathrm{d}(2N) = K, \tag{7}$$

where N_{f} is number of loading cycles at fracture.

3. One-level loading and two-step loadings

Proposed model must be in good agreement with the most important experimental data, i.e. with data from one-level and two-step loading experiments. The one-level loading data are the basic fatigue data. The two-step loading is a loading during which the first loading level is after defined number of loading cycles changed to the second level, proceeding until the fracture. In this case, considerable differences from linear hypothesis are found depending on loading sequence and loading level ratio [3, 5]. The values of three unknown constants α, β and K follow from requirement of the best quantitative agreement with experimental data.

In the one-level loading case, the quantity $\Delta V(2N)$ is constant during the whole loading, $\Delta V(2N) = \Delta V$. Then, substituting into (7), it is obtained

$$\int_0^{2N_\mathrm{f}} \left(\int_0^{2N} \Delta V^\beta \mathrm{d}(2h) \right) \Delta V^\alpha \mathrm{d}(2N) = K, \tag{8}$$

and after some arrangements

$$\Delta V = (2K)^{-\eta/2} (2N_\mathrm{f})^\eta, \tag{9}$$

where

$$\eta = -2/(\beta + \alpha). \tag{10}$$

Equation (9) is the well-known equation of fatigue-life curves [6, 7]. Then, it can be stated that I_hc and H chosen according to (2) and (4) satisfy the requirement of both, good qualitative and quantitative approximations of fatigue-lives at one--level loading. The constant K follows directly from the equation (9), and relation (10) is the first one for determination of constants α and β.

In the two-step loading case, the function of damage controlling quantity $\Delta V(2N)$ has constant values ΔV_i, $i = 1, 2$, during n_i cycles on each loading level in the loading history. It holds

$$n_i = N_i - N_{i-1}, \tag{11}$$

where N_i denotes the number of cycles at the end of the i-th loading level. Applying this to (7) it is obtained

$$\begin{aligned} &\int_0^{2N_1} \left(\int_0^{2N} \Delta V_1^\beta \mathrm{d}(2h) \right) \Delta V_1^\alpha \mathrm{d}(2N) + \\ &+ \int_{2N_1}^{2N_2} \left(\int_0^{2N_1} \Delta V_1^\beta \mathrm{d}(2h) + \int_{2N_1}^{2N} \Delta V_2^\beta \mathrm{d}(2h) \right) \Delta V_2^\alpha \mathrm{d}(2N) = K \end{aligned} \tag{12}$$

and using relation (9), after some arrangements, final expression of our hypothesis for two-step loading is obtained

$$\nu_1^2 + 2\varphi_{21}^{\vartheta}\nu_2\nu_1 + \nu_2^2 = 1, \tag{13}$$

where ν_i are cycle ratios defined by

$$\nu_i = \frac{n_i}{N_{\mathrm{f}i}}, \tag{14}$$

where $N_{\mathrm{f}i}$ is the fatigue life at one-level test with damage controlling quantity ΔV_i,

$$\varphi_{ij} = \frac{N_{\mathrm{f}i}}{N_{\mathrm{f}j}}, \tag{15}$$

and

$$\vartheta = \frac{\beta - \alpha}{\beta + \alpha} \tag{16}$$

is constant.

To gain a concrete value of constant ϑ, an experiment was performed on computer-controlled electrohydraulic test system. Plain circular specimens were made from common structural steel ($C = 0.421\%$) and tested in strain-controlled loop with zero mean stress. Nine specimens were tested on three constant strain levels giving average fatigue lives $N_{\mathrm{f}i} = 28300$, 101000 and 1060000. Results of 16 two-step tests are presented in Fig. 1. Each point represents average value of two tests, which differ from each other only by half-cycle of transition between loading levels (tension-compression). The kind of the half-cycle of transition has not resulted in greater effect on fatigue lives. On the whole, our experiment confirmed known fatigue properties of materials at this type of loading.

The value of ϑ is given by the slope of regression line through the origin in the co-ordinate system $(\log\varphi_{21}, \log[(1 - \nu_1^2 - \nu_2^2)/2\nu_1\nu_2])$, which follows from Eq. (13). Excluding the two most deviated points (lying under the line of linear hypothesis in the L-H straining case) in Fig. 1, ϑ has the value 0.26 in our case.

In Fig. 2, it can be seen that our hypothesis can also take well into account more expressive loading sequence effects and is in satisfactory qualitative agreement with well-known experimental data [3]. It can be stated, that proposed model can also satisfactorily approximate experimental data in two-step loading case. Let us still note that according to measurements of behaviour of many materials [3], it can be supposed that one value of constant ϑ holding for all materials and loading conditions can be found. In present work, however, this question will be left open.

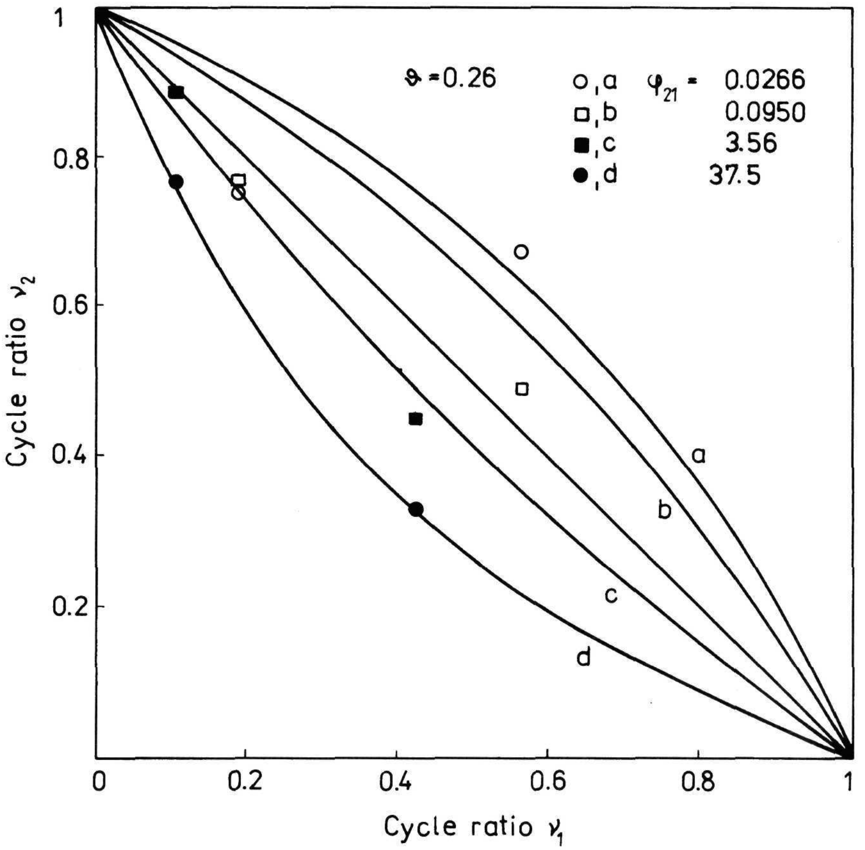

Fig. 1. Experimental points and computed curves of two-step test.

In our further considerations it is always supposed

$$\vartheta \geq 0, \tag{17}$$

which follows not only from our experiment, but also from the well-known fact that loading of H-L type results in shorter fatigue lives than the one of L-H type.

Knowing ϑ (16) and η (10), constants α and β can be determined

$$\alpha = -(1 - \vartheta)/\eta, \tag{18}$$

$$\beta = -(1 + \vartheta)/\eta. \tag{19}$$

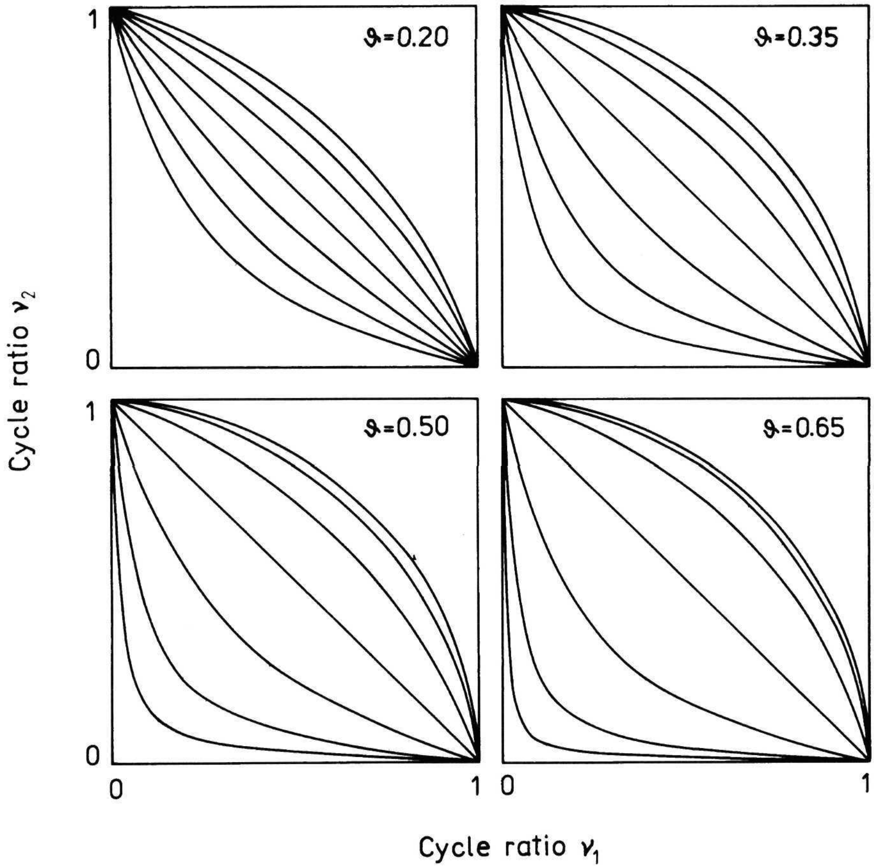

Fig. 2. Influence of exponent ϑ in the two-step loading case for $\varphi_{21} = 1000, 100, 10, 1, 0.1, 0.01, 0.001$ (for the curves from the left to the right).

Considering that only negative values of η,

$$\eta < 0, \tag{20}$$

have physical meaning, it follows from (17) and (19) that condition (5) is always satisfied. From (18) and (20), it follows that condition (3) is satisfied only when

$$\vartheta < 1. \tag{21}$$

So, inequality (21) represents the second limit for the constant ϑ.

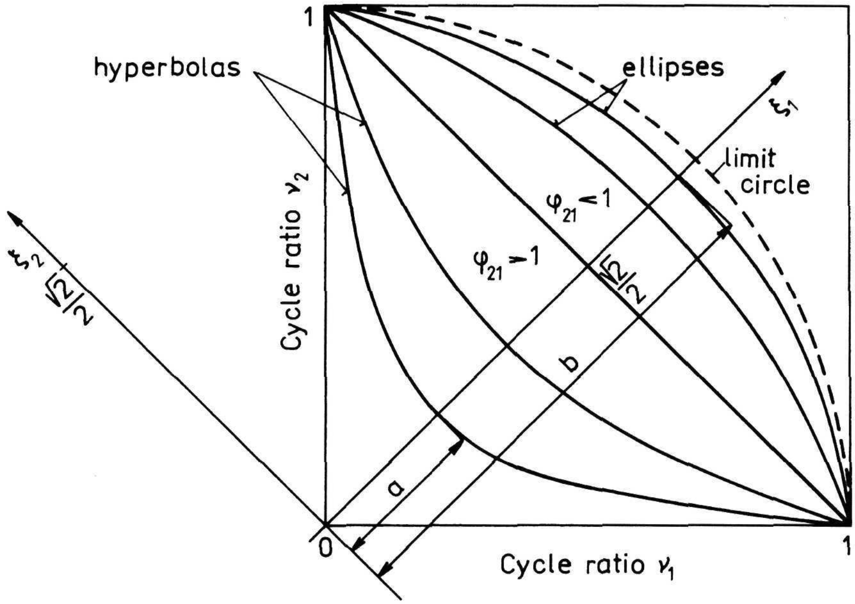

Fig. 3. Analysis of hypothesis in the two-step loading case.

Finally, let us emphasize that the Eq. (13) requires knowledge of only one constant, determined directly from experimental data.

4. Analysis

For $\varphi_{21} < 1$, Eq. (13) represents the equation of an ellipse with foci on the ξ_2--axis of co-ordinate system $\left(\xi_1 = \frac{\sqrt{2}}{2}(\nu_1 + \nu_2),\ \xi_2 = \frac{\sqrt{2}}{2}(-\nu_1 + \nu_2)\right)$ (see Fig. 3). Magnitude of its minor semi-axis b, $b = (1 + \varphi_{21}^{\vartheta})^{-1/2}$, increases with decreasing φ_{21}^{ϑ}. For $\varphi_{21} > 1$, (13) is the equation of a hyperbola with foci on the ξ_1-axis. Magnitude of its major semi-axis a, $a = (1 + \varphi_{21}^{\vartheta})^{-1/2}$, decreases with increasing φ_{21}^{ϑ}. For $\varphi_{21}^{\vartheta} = 1$, (13) reduces to the equation of a line. All three curves pass in co-ordinate system (ξ_1, ξ_2) through the points $(\sqrt{2}/2, \sqrt{2}/2)$, and $(\sqrt{2}/2, -\sqrt{2}/2)$, i.e. in co-ordinate system (ν_1, ν_2) through the points (0,1) and (1,0).

If $\vartheta = 0$, Eq. (13) represents the equation of linear hypothesis, which does not take into account loading sequence effects (Fig. 2). As ϑ increases, the deviation of hyperbolas and ellipses from the line representing linear hypothesis increases. So, parameter ϑ represents measure of sensitivity of material to the loading sequence or to interactions of loading levels.

In Fig. 2, it can be seen that the deviation of hyperbolas defined by φ_{21} from the line representing linear hypothesis is greater than the deviation of corresponding ellipses defined by φ_{21}^{-1}. For all ϑ and all φ_{21}, it can be shown that the average value of major semi-axis of hyperbola and of minor semi-axis of the corresponding ellipse is less than or equal to $\sqrt{2}/2$, which is the value of ξ_1 corresponding to the linear hypothesis (Fig. 3). Thus, in two-step loading, our hypothesis is more emphasizing the fatigue lives lower in respect to the linear hypothesis (H-L loading sequence) than the higher ones (L-H loading sequence). Because of this fact, our hypothesis need not be always in a full quantitative agreement with experimental data in L-H case. However, from the practical point of view, it is important that our life estimations are conservative.

For given ratio ν_1/ν_2 and sum $\nu_1 + \nu_2$, the left side of Eq. (13) decreases with decreasing φ_{21}. Because (13) must be satisfied, the sum $\nu_1 + \nu_2$ must increase for an arbitrarily given ν_1/ν_2. In a limit case, when φ_{21} approaches zero, Eq. (13) is reduced to

$$\nu_1^2 + \nu_2^2 = 1. \tag{22}$$

Equation (22) is the equation of unit circle (Fig. 3) representing maximum values of $\nu_1 + \nu_2$, which can be evaluated for given ν_1/ν_2 according to our hypothesis in two-step loading case.

5. Multilevel loading

It is supposed that transient parts of loading half-cycles between neighbouring blocks in the case of l loading levels can be neglected. Then, similarly as in the two-step loading case, the final expression for our hypothesis can be derived

$$\boldsymbol{\nu}'\mathbf{O}\boldsymbol{\nu} = 1, \tag{23}$$

where

$$\mathbf{O} = [o_{ij}] \tag{24}$$

is symmetric matrix, for entries of which it holds

$$o_{ij} = o_{ji} = \varphi_{ij}^{\vartheta}, \quad i = 1, 2, \ldots, l, \quad j = 1, 2, \ldots, i \tag{25}$$

and $\boldsymbol{\nu}$ is vector

$$\boldsymbol{\nu} = (\nu_1, \nu_2, \ldots, \nu_l)', \tag{26}$$

where $'$ denotes transposition and ν_i and φ_{ij} are given by (14) and (15), respectively.

In respect to the linear hypothesis, matrix $\mathbf{O}$ brings that and only that new information which characterises the influence of loading sequence (order) or history and interactions of loading levels. Vector $\boldsymbol{\nu}$ contains that and only that information, which is considered in the linear hypothesis. From the mathematical point of view, left side of (23) represents a quadratic form [8]. From the geometrical point of view, Eq. (23) is the equation of a quadratic surface centred at origin in l-dimensional space. Quadratic forms are well analysable and their mathematical theory is well worked up. So, the consequence of our hypothesis simplicity is the possibility of utilising the theory of quadratic forms in the fatigue damage analysis, too.

Finally, let us note that Eq. (23) is suitable for analysis, but it is not quite suitable for fatigue life evaluation. Effective algorithm for fatigue life evaluation can be simply derived from Eq. (23). Its description is beyond the frame of this work, however. In any case, use of our hypothesis is without special numerical problems such as reported by Halford [9].

6. Conclusions

1. Phenomenological fatigue damage hypothesis based on idea of immediate and history damage factors taking into account loading sequence effects is formulated.

2. Hypothesis is simple, synthetic and transparent and has well analysable final mathematical relation.

3. Hypothesis satisfactorily approximates basic fatigue-life curves and fatigue lives from two-step loading experiments.

4. Only one new parameter representing measure of sensitivity of material to the loading sequence or to interactions of loading levels is introduced. This parameter is determined by the two-step loading tests.

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