EFFECTIVE THERMAL CONDUCTIVITY OF FIBROUS COMPOSITE MATERIALS

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On the basis of mean field approximation method, the formula for the longitudinal and transverse effective thermal conductivity of composite with coated fibres and the relation for effective thermal conductivity of the coated fibre of the n^{th} component of fibre composite are derived. In the case of binary system, the dependence of the transverse effective thermal conductivity on the ratio of the thermal conductivity of metal matrix and effective thermal conductivity of coated fibres is defined. When this ratio is larger than 1, the effective thermal conductivity may be expressed by the rule of mixture. It is shown that at a certain value of the area fraction (percolation threshold) the percolation phase transition occurs.

K e y w o r d s: thermal conductivity, fibre composite, percolation analysis

EFEKTÍVNA TEPELNÁ VODIVOSŤ VLÁKNITÝCH KOMPOZITNÝCH MATERIÁLOV

Na základe metódy stredného poľa sa odvodil vzťah pre longitudinálnu a transverzálnu efektívnu tepelnú vodivosť vláknitých kompozitných materiálov s povrchovo upravenými vláknami. Ďalej sa odvodil vzťah pre efektívnu tepelnú vodivosť povrchovo upraveného vlákna n-tého komponentu vláknitého kompozitu. Pri binárnom systéme sa našla závislosť transverzálnej efektívnej tepelnej vodivosti vláknitého kompozitného materiálu od pomeru tepelnej vodivosti kovovej matrice k efektívnej tepelnej vodivosti povrchovo upraveného vlákna. Ak tento pomer je väčší ako jeden, potom vzťah pre efektívnu tepelnú vodivosť vláknitého kompozitu sa môže vyjadriť pomocou zmiešavacieho pravidla. Nakoniec sa ukázalo, že pri určitej hodnote plošného zlomku (prah perkolácie) prebieha perkolačný fázový prechod.

1. Introduction

According to applications the fibre-reinforced metal matrix composite should fulfill several requirements, e.g. good mechanical properties, very good thermal conductivity, low density, and tailorable coefficient of thermal expansion. Good

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mechanical properties can be achieved by enhancing the cohesive strength between fibres and metal matrix. Increasing inter-facial adhesion strength is generally carried out in two ways: by forcing fibre reaction with the metal matrix or by deposition of both Fe and Ni layers on the copper coated carbon fibres (chemical or dissolution bonding). All the mentioned properties of the fibrous composites are important in the applications where especially thermal conductivity plays large role in solving the problems of heat dissipation. The heat is produced by an integrated circuit and must be removed using a new generation of heat sink material. The carbon fibre-reinforced copper matrix composite is a candidate for applications, e.g. packaging for high voltage chips, cooling plates for microwaves and heat sinks. The physical properties and manufacturing of fibrous composites are in detail described in [1–14].

The aim of this paper is to derive the relation for effective thermal conductivity of fibrous composite materials where fibres are coated by certain suitable material for improving mechanical and thermo-physical properties. Generally, the fibrous composites are regarded as ones consisting of various kinds of coated fibres. The individual kind of coated fibres will further be called as a component. The fibrous composites may have several types of fibre orientations. In this paper we will consider unidirectional composites where all kinds of fibres are parallel and go through the whole sample.

Generally the fibrous composite material at the sub-macroscopic level (on the length scale of linear dimension of fibres) is heterogeneous because it is composed of certain components which, on the one hand, are spatially separated from each other and, on the other hand, are randomly distributed over the whole cross-section of the sample. Due to this randomness the local physical quantities at the sub-macroscopic level are not only dependent on space coordinates but they are also random quantities. For this reason the heat conduction on the sub-macroscopic level is described by the phenomenological stochastic heat conduction equation. However at the macroscopic level fibrous composite usually is homogeneous but anisotropic and it may be characterized by the effective tensor thermal conductivity which is independent on space coordinates.

For derivation of the relation for effective tensor thermal conductivity one uses the phenomenological heat conduction equation. This is justified only if the linear dimensions of fibres are much larger than the mean free path of heat carriers which participate on the transport of energy, but, on the other hand, they have to be much smaller with respect to the macroscopic scale.

Manufacturing of fibrous composite, it is very important to know how the effective parameters depend on the structure of fibrous composite at the sub-macroscopic level and also on the properties of the individual components of fibrous composite. This information is very important especially for technologists. This is the main reason why we derive the relation for the effective tensor thermal conductivity. The statistics of the structure of the fibrous composite at the sub--macroscopic level is usually unknown and what we only know are the area fractions which are known from the manufacturing process. If the fibrous composite at the macroscopic level is a homogeneous one, we can use the following **Assumption**: *The probability of the occupation of the* n^{th} *component on a certain place in the plane perpendicular to fibres is equal to its areal fraction.* From this point of view one deals with a 2-dimensional case. The derivation of the relation for the effective tensor thermal conductivity is connected with mathematical difficulties and, therefore, one is obliged to use approximate methods. One of these approximate methods is the mean field approximation method (MFAM), which is used in this article. The MFAM is based on the idea that a randomly chosen fibre of the n^{th} component, characterized by the tensor thermal conductivity $\boldsymbol{\lambda}_n$, is submerged into an unlimited effective medium, which is characterized by the effective tensor thermal conductivity $\boldsymbol{\lambda}_{\mathrm{eff}}$. The MFAM assumes that by putting the fibre of the n^{th} component into the effective medium its properties do not change. It can be shown that in the case of weak inhomogeneity the MFAM may give results of sufficient accuracy [15]. The further information about the statistics of composites is given in [16].

2. Effective tensor thermal conductivity of fibrous composite materials

We will consider the continuous fibre-reinforced matrix composite. The fibres of the n^{th} component may be covered with the surface layer characterized by the tensor thermal conductivity $\boldsymbol{\lambda}_{\mathrm{s}n}$. We introduce the coordinate system where the y and z axes are perpendicular to fibres and x axis is parallel to the fibres. Then the tensor thermal conductivities have the following form:

$$\boldsymbol{\lambda}_n = \lambda_{n||}\,\boldsymbol{ii} + \lambda_{n\perp}(\boldsymbol{jj} + \boldsymbol{kk})$$

is the tensor thermal conductivity of the n^{th} component,

$$\boldsymbol{\lambda}_{\mathrm{s}n} = \lambda_{\mathrm{s}n||}\,\boldsymbol{ii} + \lambda_{\mathrm{s}n\perp}(\boldsymbol{jj} + \boldsymbol{kk})$$

is the tensor thermal conductivity of the surface layer of the n^{th} component,

$$\boldsymbol{\lambda}_{\mathrm{eff}} = \lambda_{\mathrm{eff}}\,\boldsymbol{ii} + \lambda_{\mathrm{eff}}(\boldsymbol{jj} + \boldsymbol{kk})$$

is the effective tensor thermal conductivity of the fibrous composite. $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ are the unit vectors in the directions of x, y and z axis, respectively.

The fibrous composite at the macroscopic level is anisotropic also in the case when the fibres and the surface layers are isotropic. The stationary heat equation in the effective medium due to anisotropy is as follows

$$\lambda_{\mathrm{eff}||}\frac{\partial^2\langle T\rangle}{\partial x^2}+\lambda_{\mathrm{eff}\perp}\left(\frac{\partial^2\langle T\rangle}{\partial y^2}+\frac{\partial^2\langle T\rangle}{\partial z^2}\right)=0, \tag{1}$$

where $\langle T\rangle$ is the average temperature due to the macroscopic character of measuring instrument. The experimentalist observes the temperature averaged over large number of fibres and, therefore, $T_{\mathrm{exp}}=\langle T\rangle$. The symbol $\langle\rangle$ means the average value over the representative volume element. The expression for effective tensor thermal conductivity will be found in two steps. At the first step we will derive the effective thermal conductivity $\lambda_{\mathrm{eff}\perp}$ and at the second one the $\lambda_{\mathrm{eff}||}$.

2.1 Perpendicular component of tensor thermal conductivity $\lambda_{\mathrm{eff}\perp}$

At first, as in the case of a particulate composite [16], we will consider the case of fibres without coating. We find the effective thermal conductivity $\lambda_{\mathrm{eff}\perp}$ and then we focus our attention to the case when the fibres of the certain components are coated. This case represents the main contribution of this paper.

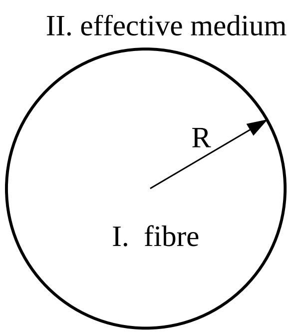

Fig. 1. Graphical representation of cross-section of uncoated fibre in effective medium.

We will consider two regions (Fig. 1). Due to the circular symmetry the temperature $T(y,z)$ is only a function of $r=\sqrt{y^2+z^2}$. The stationary heat equation has the form:
in region I (fibre)

$$\Delta T^{(\mathrm{I})}=0 \tag{2}$$

and in the region II (effective medium)

$$\Delta T^{(\mathrm{II})}=0. \tag{3}$$

The solution of Eqs. (2) and (3) considering the condition in infinity

$$\lim_{r\to\infty}\nabla T^{(\mathrm{II})}=-\boldsymbol{E}$$

is the following:
in region I

$$T^{(\mathrm{I})}=-B\boldsymbol{E}\cdot\boldsymbol{r} \tag{4}$$

and in region II

$$T^{(\mathrm{II})} = -\boldsymbol{E} \cdot \boldsymbol{r} + \frac{C}{r^2} \boldsymbol{E} \cdot \boldsymbol{r}. \tag{5}$$

The constants B and C are determined from the boundary conditions

$$T^{(\mathrm{I})}(R) = T^{(\mathrm{II})}(R), \tag{6}$$

$$-\lambda_{n\perp} \nabla T^{(\mathrm{I})} \cdot \boldsymbol{r}_0 = -\lambda_{\mathrm{eff}\perp} \nabla T^{(\mathrm{I})} \cdot \boldsymbol{r}_0, \tag{7}$$

where $\boldsymbol{r}_0$ is the unit vector perpendicular to fibre. Condition (7) expresses the equality of the heat flow densities in the radial direction. From boundary conditions (6) and (7) it follows

$$B = \frac{1}{1 + \dfrac{\lambda_{n\perp} - \lambda_{\mathrm{eff}\perp}}{2\lambda_{\mathrm{eff}\perp}}} \tag{8}$$

and

$$\nabla T^{(\mathrm{I})} = -\frac{1}{1 + \dfrac{\lambda_{n\perp} - \lambda_{\mathrm{eff}\perp}}{2\lambda_{\mathrm{eff}\perp}}} \boldsymbol{E}, \tag{9}$$

$$\boldsymbol{q} = -\frac{\lambda_{n\perp}}{1 + \dfrac{\lambda_{n\perp} - \lambda_{\mathrm{eff}\perp}}{2\lambda_{\mathrm{eff}\perp}}} \boldsymbol{E} = -\lambda_{n\perp} \nabla T^{(\mathrm{I})}, \tag{10}$$

where $\boldsymbol{q}$ is the transverse heat flow density in the fibre of the n^{th} component.

Using the **Assumption**, we can average relations (9) and (10). After averaging we obtain the relations

$$\nabla \langle T^{(\mathrm{I})} \rangle = -\sum_{n=1}^{N} c_n \frac{1}{1 + \dfrac{\lambda_{n\perp} - \lambda_{\mathrm{eff}\perp}}{2\lambda_{\mathrm{eff}\perp}}} \boldsymbol{E} \tag{11}$$

and

$$\langle \boldsymbol{q} \rangle = -\sum_{n=1}^{N} c_n \frac{\lambda_{n\perp}}{1 + \dfrac{\lambda_{n\perp} - \lambda_{\mathrm{eff}\perp}}{2\lambda_{\mathrm{eff}\perp}}} \boldsymbol{E}, \tag{12}$$

where c_n is the area fraction of the n^{th} component in the plane perpendicular to the fibres.

Relation (12) can be written in the form

$$\langle \boldsymbol{q} \rangle = -\sum_{n=1}^{N} c_n \frac{\lambda_{n\perp} - \lambda_{\mathrm{eff}\perp}}{1 + \dfrac{\lambda_{n\perp} - \lambda_{\mathrm{eff}\perp}}{2\lambda_{\mathrm{eff}\perp}}} \boldsymbol{E} - \lambda_{\mathrm{eff}\perp} \nabla \langle T^{(\mathrm{I})} \rangle. \tag{13}$$

On the other hand, one can write

$$\langle \boldsymbol{q} \rangle = -\lambda_{\mathrm{eff}\perp} \nabla \langle T^{(\mathrm{I})} \rangle. \tag{14}$$

From (13) and (14) it immediately follows

$$\sum_{n=1}^{N} c_n \frac{\lambda_{n\perp} - \lambda_{\mathrm{eff}\perp}}{1 + \dfrac{\lambda_{n\perp} - \lambda_{\mathrm{eff}\perp}}{2\lambda_{\mathrm{eff}\perp}}} = 0. \tag{15}$$

For the interpretation of the quantity $\boldsymbol{E}$ (integration constant) we can choose $\boldsymbol{E} = -\nabla \langle T^{(\mathrm{I})} \rangle$. By substituting this into (11) and (12) one obtains

$$\sum_{n=1}^{N} c_n \frac{1}{1 + \dfrac{\lambda_{n\perp} - \lambda_{\mathrm{eff}\perp}}{2\lambda_{\mathrm{eff}\perp}}} = 1 \tag{16}$$

and

$$\langle \boldsymbol{q} \rangle = -\sum_{n=1}^{N} c_n \frac{\lambda_{n\perp}}{1 + \dfrac{\lambda_{n\perp} - \lambda_{\mathrm{eff}\perp}}{2\lambda_{\mathrm{eff}\perp}}} \nabla \langle T^{(\mathrm{I})} \rangle. \tag{17}$$

From (14) and (17) it immediately follows

$$\lambda_{\mathrm{eff}\perp} = \sum_{n=1}^{N} c_n \frac{\lambda_{n\perp}}{1 + \dfrac{\lambda_{n\perp} - \lambda_{\mathrm{eff}\perp}}{2\lambda_{\mathrm{eff}\perp}}}. \tag{18}$$

It can be easily shown that relation (18) follows from (15) and (16). The effective thermal conductivity $\lambda_{\mathrm{eff}\perp}$ can be calculated either from relation (15) or from (16).

Now, we will consider the case when the fibres are coated. In this case we have three regions (Fig. 2) where we use MFAM. In the first region there is the fibre of the n^{th} component with thermal conductivity $\lambda_{n\perp}$ and with the radius R_{1n}. In the second region there is the surface layer with the thermal conductivity $\lambda_{\mathrm{s}n\perp}$ and with the radius of the surface layer together with fibre R_{2n}. The third region is the effective medium. The stationary heat equation has the following form:

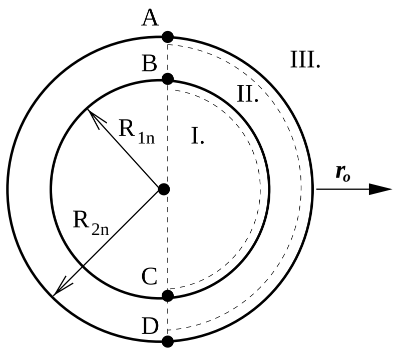

Fig. 2. Graphical representation of cross-section of coated fibre in effective medium.

in the region I (fibre)

$$\Delta T^{(\mathrm{I})} = 0, \tag{19}$$

in region II (surface layer)

$$\Delta T^{(\mathrm{II})} = 0, \tag{20}$$

in region III (effective medium)

$$\Delta T^{(\mathrm{III})} = 0. \tag{21}$$

The solutions of Eqs. (19), (20) and (21) are expressed by the relations:

in the region I

$$T^{(\mathrm{I})} = -B\boldsymbol{E}\cdot\boldsymbol{r}, \tag{22}$$

in the region II

$$T^{(\mathrm{II})} = -C\boldsymbol{E}\cdot\boldsymbol{r} + \frac{D}{r^2}\boldsymbol{E}\cdot\boldsymbol{r}, \tag{23}$$

in the region III

$$T^{(\mathrm{III})} = -\boldsymbol{E}\cdot\boldsymbol{r} + \frac{F}{r^2}\boldsymbol{E}\cdot\boldsymbol{r}. \tag{24}$$

Relation (24) fulfills the boundary condition in infinity

$$\lim_{r\to\infty} \nabla T^{(\mathrm{III})} = -\boldsymbol{E}.$$

The constants B, C, D, and F are determined from the boundary conditions:
at $r = R_{1n}$

$$T^{(\mathrm{I})}(R_{1n}) = T^{(\mathrm{II})}(R_{1n}), \tag{25}$$

$$-\lambda_{n\perp}\nabla T^{(\mathrm{I})}(R_{1n})\cdot\boldsymbol{r}_0 = -\lambda_{\mathrm{sn}\perp}\nabla T^{(\mathrm{II})}(R_{1n})\cdot\boldsymbol{r}_0, \tag{26}$$

and at $r = R_{2n}$

$$T^{(\mathrm{II})}(R_{2n}) = T^{(\mathrm{III})}(R_{2n}), \tag{27}$$

$$-\lambda_{\mathrm{sn}\perp}\nabla T^{(\mathrm{II})}(R_{2n})\cdot\boldsymbol{r}_0 = -\lambda_{\mathrm{eff}\perp}\nabla T^{(\mathrm{III})}(R_{2n})\cdot\boldsymbol{r}_0. \tag{28}$$

Conditions (26) and (28) express the continuity of the heat flow densities in the radial direction. By substituting (22–24) into (25–28) one obtains

$$B = C - \frac{D}{R_{1n}^2}, \tag{29}$$

$$C - \frac{D}{R_{2n}^2} = 1 - \frac{F}{R_{2n}^2}, \tag{30}$$

$$\lambda_{n\perp} B = \lambda_{\mathrm{s}n\perp}(C + \frac{D}{R_{1n}^2}), \tag{31}$$

$$\lambda_{\mathrm{s}n\perp}\left(C + \frac{D}{R_{2n}^2}\right) = \lambda_{\mathrm{eff}\perp}\left(1 - \frac{F}{R_{2n}^2}\right). \tag{32}$$

From Eqs. (29) and (31) it follows

$$C = \frac{\lambda_{n\perp} + \lambda_{\mathrm{s}n\perp}}{2\lambda_{\mathrm{s}n\perp}} B \tag{33}$$

and

$$\frac{D}{R_{1n}^2} = \frac{\lambda_{n\perp} - \lambda_{\mathrm{s}n\perp}}{2\lambda_{\mathrm{s}n\perp}} B. \tag{34}$$

From Eqs. (30) and (32) one can obtain the following relation:

$$\lambda_{\mathrm{eff}\perp}\left(C - \frac{D}{R_{2n}^2}\right) + \lambda_{\mathrm{s}n\perp}\left(C + \frac{D}{R_{1n}^2}\right) = 2\lambda_{\mathrm{eff}\perp}. \tag{35}$$

By substituting (33) and (34) into (35) one obtains

$$B = 4\frac{\lambda_{\mathrm{eff}\perp}\lambda_{\mathrm{s}n\perp}}{\lambda_{n\perp}} \frac{1}{\lambda_{\mathrm{eff}\perp}[1 + \gamma_n - \alpha_n(1 - \gamma_n)] + \lambda_{\mathrm{s}n\perp}[1 + \gamma_n + \alpha_n(1 - \gamma_n)]}, \tag{36}$$

where $\gamma_n = \dfrac{\lambda_{\mathrm{s}n\perp}}{\lambda_{n\perp}}$ and $\alpha_n = \left(\dfrac{R_{1n}}{R_{2n}}\right)^2$. With the help of relations (33) and (34) one can write

$$\lambda_{\mathrm{eff}\perp}\left(C - \frac{D}{R_{2n}^2}\right) = \frac{\lambda_{\mathrm{eff}\perp}\lambda_{n\perp}}{2\lambda_{\mathrm{s}n\perp}}[1 + \gamma_n - \alpha_n(1 - \gamma_n)]B \tag{37}$$

and

$$\lambda_{\mathrm{s}n\perp}\left(C + \frac{D}{R_{2n}^2}\right) = \frac{\lambda_{n\perp}}{2}[1 + \gamma_n + \alpha_n(1 - \gamma_n)]B. \tag{38}$$

In the case of stationary heat conduction, we can write $\nabla \cdot \boldsymbol{q} = 0$. Due to this fact the relation

$$\oint \boldsymbol{q} \cdot \mathrm{d}\boldsymbol{s} = 0 \tag{39}$$

for arbitrary closed curve is valid, where $\mathrm{d}\boldsymbol{s} = \mathrm{d}s\, \boldsymbol{r}_0$ is perpendicular to closed curve and is oriented outwards. The heat flow through the $\overline{AD}$ (Fig. 2) is expressed by the relation

$$Q_n = -\int_A^D \boldsymbol{q}_n \cdot \mathrm{d}\boldsymbol{s}. \tag{40}$$

Using relation (39) one can write

$$Q_n = \int_D^A \boldsymbol{q}_n \cdot \mathrm{d}\boldsymbol{s}. \tag{41}$$

From (23) it follows

$$\boldsymbol{q}_n = \lambda_{\mathrm{s}n\perp} \left[C\boldsymbol{E} - \frac{D}{R_{2n}^2}\boldsymbol{E} + 2\frac{D}{R_{2n}^2}\boldsymbol{E} \cdot \boldsymbol{r}_0\, \boldsymbol{r}_0 \right]. \tag{42}$$

By substituting (42) into (41) we obtain

$$\begin{aligned} Q_n &= \lambda_{\mathrm{s}n\perp} \left[C + \frac{D}{R_{2n}^2} \right] \int_D^A \boldsymbol{E} \cdot \boldsymbol{r}_0 \mathrm{d}s = \\ &= \lambda_{\mathrm{s}n\perp} \left[C + \frac{D}{R_{2n}^2} \right] \int_{-\pi/2}^{+\pi/2} E\, R_{2n} \cos\varphi \mathrm{d}\varphi = \\ &= \lambda_{\mathrm{s}n\perp} \left[C - \frac{D}{R_{2n}^2} \right] 2ER_{2n}. \end{aligned} \tag{43}$$

According to (43) the heat flow density is expressed by the relation

$$\overline{q}_n = \frac{Q_n}{2R_{2n}} = \lambda_{\mathrm{s}n\perp} \left[C + \frac{D}{R_{2n}^2} \right] E. \tag{44}$$

The direction of $\boldsymbol{q}$ is parallel to $\boldsymbol{E}$ if we orient y-axis parallel to $\boldsymbol{E}$, so we can write

$$\overline{\boldsymbol{q}}_n = \lambda_{\mathrm{s}n\perp} \left[C + \frac{D}{R_{2n}^2} \right] \boldsymbol{E}. \tag{45}$$

By substituting (36) and (38) into (45) one obtains

$$\overline{\boldsymbol{q}}_n = \frac{\lambda_{n\perp}}{2}[1+\gamma_n+\alpha_n(1-\gamma_n)]4\frac{\lambda_{\mathrm{eff}\perp}\lambda_{\mathrm{s}n\perp}}{\lambda_{n\perp}} \frac{1}{\lambda_{\mathrm{eff}\perp}[1+\gamma_n-\alpha_n(1-\gamma_n)]+\lambda_{\mathrm{s}n\perp}[1+\gamma_n+\alpha_n(1-\gamma_n)]}. \tag{46}$$

After some arrangement we obtain

$$\overline{\boldsymbol{q}}_n = \frac{\lambda^*_{n\perp}}{1+\dfrac{\lambda^*_{n\perp}-\lambda_{\mathrm{eff}\perp}}{2\lambda_{\mathrm{eff}\perp}}}\boldsymbol{E}, \tag{47}$$

where

$$\lambda^*_{n\perp} = \lambda_{\mathrm{s}n\perp}\frac{1+\gamma_n+\alpha_n(1-\gamma_n)}{1+\gamma_n-\alpha_n(1-\gamma_n)}. \tag{48}$$

Comparison of relations (17) and (47) shows that $\lambda^*_{n\perp}$ can be considered as the effective thermal conductivity of the coated fibre of the n^{th} component. In Fig. 3 the dependence of $\dfrac{\lambda^*_{n\perp}}{\lambda_{n\perp}}$ on the γ_n for the various α_n is shown. Finally, it is necessary to present the physical meaning of the quantity $\boldsymbol{E}$. For this reason we calculate the average value of $\nabla T^{(\mathrm{II})}$ within two semicircles AD and BC. Using relation (23) one can write

$$\overline{\nabla T^{(\mathrm{II})}} = \frac{1}{\pi/2(R^2_{2n}-R^2_{1n})}\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}\int_{R_{1n}}^{R_{2n}}\left[-C\boldsymbol{E}+\frac{D}{r^2}\boldsymbol{E}-2\frac{D}{r^2}\boldsymbol{E}\cdot\boldsymbol{r}_0\boldsymbol{r}_0\right]r\mathrm{d}\varphi\mathrm{d}r = -C\boldsymbol{E}, \tag{49}$$

where we used $\boldsymbol{E}=E\boldsymbol{j}$ and $\boldsymbol{r}_0=\boldsymbol{j}\cos\varphi+\boldsymbol{k}\sin\varphi$. The average value of the $\nabla T^{(\mathrm{I})}$ within the area of $\overline{BC}$ and the semicircle of radius R_{1n} is expressed by the relation

$$\overline{\nabla T^{(\mathrm{I})}} = \frac{1}{\dfrac{\pi}{2}R^2_{1n}}\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}\int_{0}^{R_{1n}}B\boldsymbol{E}r\mathrm{d}\varphi\mathrm{d}r = -B\boldsymbol{E}. \tag{50}$$

According to relations (49) and (50), the average value within the area of $\overline{AD}$ and the semicircle of the radius R_{2n} is defined as follows:

$$\overline{\nabla T} = \frac{\dfrac{\pi}{2}R^2_{1n}\overline{\nabla T^{(\mathrm{I})}}+\dfrac{\pi}{2}(R^2_{2n}-R^2_{1n})\overline{\nabla T^{(\mathrm{II})}}}{\dfrac{\pi}{2}R^2_{2n}} = \frac{-R^2_{1n}B-(R^2_{2n}-R^2_{1n})C}{R^2_{2n}}\boldsymbol{E} = -\frac{CR^2_{1n}-D+R^2_{2n}C-R^2_{1n}C}{R^2_{2n}}\boldsymbol{E} = (C-\frac{D}{R^2_{2n}})\boldsymbol{E}, \tag{51}$$

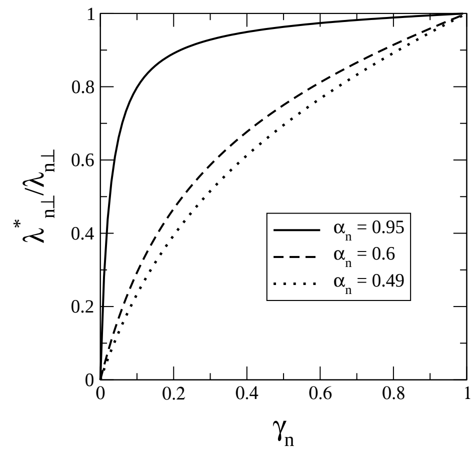

Fig. 3. Plot of $y = \dfrac{\lambda^*_{n\perp}}{\lambda_{n\perp}}$ vs. $\gamma_n = \dfrac{\lambda_{sn\perp}}{\lambda_{n\perp}}$.

where we used (29).

By inserting (36), (37) and (48) into (51) one obtains

$$\overline{\nabla T} = -\frac{1}{1 + \dfrac{\lambda^*_{n\perp} - \lambda_{\mathrm{eff}\perp}}{2\lambda_{\mathrm{eff}\perp}}}\boldsymbol{E}. \tag{52}$$

If we substitute $\lambda^*_{n\perp}$ instead of $\lambda_{n\perp}$ in relation (9), we obtain relation (52). Averaging (52) according to **Assumption** one obtains

$$\langle\overline{\nabla T}\rangle = -\sum_{n=1}^{N} c_n \frac{1}{1 + \dfrac{\lambda^*_{n\perp} - \lambda_{\mathrm{eff}\perp}}{2\lambda_{\mathrm{eff}\perp}}}\boldsymbol{E}. \tag{53}$$

Proceeding analogically as in (11–18), we obtain the following relations:

$$\boldsymbol{E} = -\langle\overline{\nabla T}\rangle, \tag{54}$$

$$\sum_{n=1}^{N} c_n \frac{1}{1 + \dfrac{\lambda^*_{n\perp} - \lambda_{\mathrm{eff}\perp}}{2\lambda_{\mathrm{eff}\perp}}} = 1, \tag{55}$$

$$\sum_{n=1}^{N} c_n \frac{\lambda^*_{n\perp} - \lambda_{\mathrm{eff}\perp}}{1 + \dfrac{\lambda^*_{n\perp} - \lambda_{\mathrm{eff}\perp}}{2\lambda_{\mathrm{eff}\perp}}} = 0. \tag{56}$$

The effective thermal conductivity can be calculated either from (55) or from (56).

If the first N_1 components consist of the coated fibres, the effective thermal conductivity can be obtained as a solution of the following equation:

$$\sum_{n=1}^{N_1} c_n \frac{\lambda^*_{n\perp} - \lambda_{\mathrm{eff}\perp}}{1 + \dfrac{\lambda^*_{n\perp} - \lambda_{\mathrm{eff}\perp}}{2\lambda_{\mathrm{eff}\perp}}} + \sum_{n=N_1+1}^{N} c_n \frac{\lambda_{n\perp} - \lambda_{\mathrm{eff}\perp}}{1 + \dfrac{\lambda_{n\perp} - \lambda_{\mathrm{eff}\perp}}{2\lambda_{\mathrm{eff}\perp}}} = 0. \tag{57}$$

The longitudinal effective thermal conductivity $\lambda_{\mathrm{eff}||}$ can be calculated very easily because all fibres are parallel, so we can directly write

$$\lambda_{\mathrm{eff}||} = \sum_{n=1}^{N} c_n \lambda^*_{n||}, \tag{58}$$

where

$$\lambda^*_{n||} = \frac{S_n \lambda_{n||} + S_{\mathrm{s}} \lambda_{\mathrm{s}n||}}{S_n + S_{\mathrm{s}}},$$

S_n is the area of the cross-section of fibre of n^{th} component, S_{s} is the area of the cross-section of the surface layer of the n^{th} component. The present method can be applied also in a more general case when the fibres are cross-plied or woven in-plane.

3. Analysis and generalization of the obtained results

Usually the fibrous composite consists of the same kind of coated fibres and of the metal matrix (binary system). The metal matrix is considered as an isotropic one. In this case the effective transverse thermal conductivity is obtained according to relation (57) by the solution of the following equation

$$c\frac{1-x}{1+x} + (1-c)\frac{r-x}{r+x} = 0, \tag{59}$$

where c is the area fraction of the coated fibres component in the plane perpendicular to the fibres, $x = \dfrac{\lambda_{\mathrm{eff}\perp}}{\lambda^*_{1\perp}}$, $r = \dfrac{\lambda_2}{\lambda^*_{1\perp}}$, λ_2 is the thermal conductivity of the metal matrix and $\lambda^*_{1\perp}$ is the effective thermal conductivity of the coated fibre.

The solution of Eq. (59) is as follows:

$$x = \{(1-r)(1-c) + \sqrt{(1-r)^2(c-\frac{1}{2})^2 + r}\}. \tag{60}$$

If $\lambda_2 = 0$, the relation (60) transforms into the form

$$x = 0; \quad c \leq \frac{1}{2} \tag{61}$$

and

$$x = 2\left(c - \frac{1}{2}\right); \quad c > \frac{1}{2}. \tag{62}$$

The area fraction $c_k = \frac{1}{2}$ is called the percolation threshold. The interpretation of relations (61) and (62) is as follows: The cross-section of the coated fibre in the plane perpendicular to the fibres has the circular form. These cross-sections of the coated fibres form clusters. In the case $c < c_k$ these clusters are separated from each other and, therefore, a sample is thermally non-conducting ($\lambda_{\mathrm{eff}\perp} = 0$). At $c = c_k$ some clusters connect themselves together and form a percolation cluster, which is spread out through the whole plane perpendicular to the fibres. From this moment $\lambda_{\mathrm{eff}\perp}$ is increasing with area fraction c. This effect is called percolation and at $c = c_k$ the percolation phase transition takes place. The detailed overview of the percolation is given in [17].

The percolation threshold c_k for different types of the 2-dimensional lattices are given in Table 1 in [16]. These values were obtained by Monte Carlo simulations on different 2-dimensional lattices. MFAM yields the value $c_k = \frac{1}{2}$, which is the same as for triangular lattice. In the further text we denote c_k as g and we will consider g as a free parameter which will be determined from the best fitting of the experimental data with the theoretical relation for the effective thermal conductivity.

From the renormalization group analysis it follows that instead of (59) the following relation is valid:

$$x = 2(c - g)^t; \qquad c > g. \tag{63}$$

It is interesting to note that the parameter t (critical index) depends only on dimensionality of the sample. For 2-dimensional case $t = T = 1.15$, but relation (63) with this value of parameter t is valid only near to the percolation threshold g.

Now we approach the generalization of relation (59) analogically as in [16]. The generalization will be done so as to fulfill the following requirements:

- For $c = 0 \quad x = \frac{\lambda_2}{\lambda^*_{1\perp}}$.
- For $c = 1 \quad x = 1$.

- For $r = 0$ relation (63) is valid with the parameter $t = T$ close to the percolation threshold g (critical domain).
- For $t = 1$ and $g = \dfrac{1}{2}$ it has to give relation (62).

The relation which fulfills the above-mentioned requirements has the following form:

$$c\frac{1 - x^{\frac{1}{t}}}{(1-g)x^{\frac{1}{t}} + g} + (1-c)\frac{r^{\frac{1}{t}} - x^{\frac{1}{t}}}{(1-g)x^{\frac{1}{t}} + gr^{\frac{1}{t}}} = 0. \quad (64)$$

The solution of Eq. (64) can be expressed in the form:

$$x = \left\{ \frac{c(1-r^{\frac{1}{t}}) + (1-g)r^{\frac{1}{t}} - g}{2(1-g)} + \sqrt{\left[\frac{c(1-r^{\frac{1}{t}}) + (1-g)r^{\frac{1}{t}} - g}{2(1-g)}\right]^2 + \frac{g}{1-g}r^{\frac{1}{t}}} \right\}^t . \quad (65)$$

For $r = 0$ one obtains

$$x = 0, \qquad c \leq g \quad (66)$$

and

$$x = \frac{(c-g)^t}{(1-g)^t}, \quad c > g. \quad (67)$$

The computer simulations on the lattices show that for increasing c the parameter t is approaching 1. From this fact it follows that t is dependent on c. The form of the function $t(c)$ may be determined from the conditions: $t = 1$ at $c = 0$ and $c = 1$; $t = T = 1.15$ at $c = g$. If we expand function $t(c)$ into power series according to c up to the quadratic term and consider the above-mentioned requirements we can write the relation

$$t(c) = 1 + \frac{T-1}{g(1-g)}c(1-c). \quad (68)$$

For graphical illustration of the difference between relations (60) and (65) the dependence of x vs. c is depicted in Fig. 4. The full line corresponds to $r = 1.2$, $t = 1$ and $g = \frac{1}{2}$, the cross one to $r = 1.2$, $g = \dfrac{1}{2}$ and $T = 1.15$. From the Fig. 4 it is evident that the difference between the both dependences may be neglected (this is valid only for $r > 1$). This behaviour of the dependences in the 2-dimensional case is different from that in the 3-dimensional case [16]. Due to this fact in the further text we will consider $t = 1$. For the illustration of the behaviour of the dependences $x = \dfrac{\lambda_{\text{eff}\perp}}{\lambda^*_{1\perp}}$ vs. c these dependences are depicted in Fig. 5 for $r < 1$ and for $r > 1$. From Fig. 5 it is evident that the curves approach to the straight

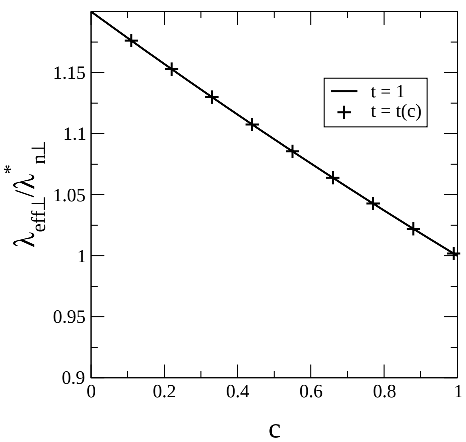

Fig. 4. Relative transverse effective thermal conductivity $x = \dfrac{\lambda_{\mathrm{eff}\perp}}{\lambda^*_{n\perp}}$ vs. area fraction of the fibres, c; $r = 1.2$; $g = 0.5$; $t(c) = 1 + (T-1)\dfrac{c(1-c)}{g(1-g)}$; $T = 1.15$.

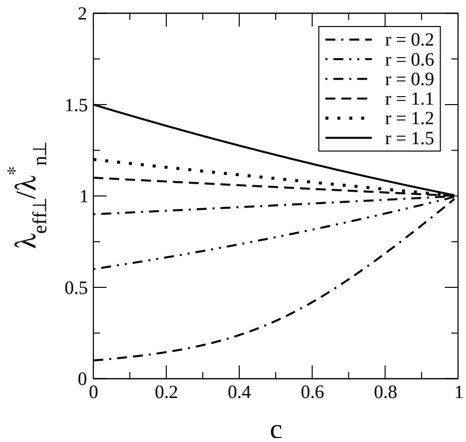

Fig. 5. Relative transverse effective thermal conductivity $x = \dfrac{\lambda_{\mathrm{eff}\perp}}{\lambda^*_{n\perp}}$ vs. area fraction of the fibres, c; $t = 1$ and $g = 0.5$.

line if r approaches to $r = 1$ or $r > 1$. If $r > 1$, one can express the dependence of $x = \dfrac{\lambda_{\mathrm{eff}\perp}}{\lambda^*_{1\perp}}$ vs. c with the sufficient accuracy by the following relation

$$x = a + bc.$$

It is evident that for $c = 0$, $x = \dfrac{\lambda_2}{\lambda^*_{1\perp}}$ and for $c = 1$, $x = 1$. Considering these facts one obtains

$$x = \frac{\lambda_2}{\lambda^*_{1\perp}} - \left(\frac{\lambda_2}{\lambda^*_{1\perp}} - 1\right) c. \tag{69}$$

Relation (69) expresses the rule of mixtures.

It is interesting to note that for $c = g = \dfrac{1}{2}$ it follows from (65) that $x = \sqrt{r}$ for the arbitrary value of t. For this case one can write $\lambda_{\mathrm{eff}\perp} = \sqrt{\lambda_2 \lambda^*_{1\perp}}$ which is the geometric average.

The sharp percolation phase transition can be observed only in the case when $r = 0$. In real applications the thermal conductivity of the metal matrix is finite and, therefore, we cannot observe the percolation phase transition. But there is a condition, when the percolation, in a certain sense, may be identified. Proceeding analogically as in [16] one can show that the second derivative of the function $x(c)$ has the maximum near the percolation threshold g. It can be shown that for $t = 1$ and $r < 1$ the relation

$$c^* = \frac{g(1+r) - r}{1 - r} \tag{70}$$

is valid. The c^* means the value at which $\frac{\mathrm{d}^2 x}{\mathrm{d}c^2}$ has its maximum. From (70) it follows: the smaller is r the better $c^* \approx g$ holds.

4. Conclusions

– The formula for the longitudinal and transverse effective thermal conductivity of fibrous composite with coated fibres of certain components was derived.

– Further, the relation for the effective thermal conductivity $\lambda_{n\perp}^*$ of the coated fibre of the n^{th} component of fibrous composite was derived.

– In the case of binary system, the formula for the transverse effective thermal conductivity of fibrous composite was generalized by introducing the parameter t which may be dependent on the area fraction c of the fibres. But the numerical calculation has shown that the parameter t is weakly dependent on c. Therefore, it is sufficient to consider $t = 1$.

– In the case of binary system an analysis of the obtained results was performed.

– The transverse effective thermal conductivity of binary system in the case when $c = g = \frac{1}{2}$ (the percolation threshold) is independent on t, and it can be expressed as a square root of the geometric average of the thermal conductivity of metal matrix λ_2 and the effective thermal conductivity $\lambda_{1\perp}^*$ of the coated fibre.

– For r near to $r = 1$ and for $r > 1$ the transverse effective thermal conductivity of binary system can be calculated with sufficient accuracy by using the rule of mixtures.

– It was shown that in the case when $\lambda_2 = 0$ and $c_k = g$ (percolation threshold), the percolation phase transition takes place.

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