YOUNG'S MODULUS CALCULATIONS FOR SYSTEMS WITH PERIODICALLY DISTRIBUTED IDENTICAL SPHEROIDAL PORES OF VARIOUS SIZE, ORIENTATION, AND SHAPE ANISOTROPY

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In the presented paper the Young's modulus for samples with periodically distributed spheroidal pores is calculated. The dependence of Young's modulus on the total porosity, size, orientation, and shape anisotropy of pores is investigated. It is shown that presence of pores flattened in the loading direction significantly reduces the Young's modulus value already at low porosity.

The Young's modulus as a function of porosity is also calculated for the system with the pore flattening dependent on the value of total porosity. The theoretical curve is compared with the experimental one for ceramics Dy_2O_3.

VÝPOČET YOUNGOVHO MODULU PRE SYSTÉMY S PERIODICKY ROZLOŽENÝMI IDENTICKÝMI SFÉROIDICKÝMI PÓRMI RÔZNEJ VEĽKOSTI, ORIENTÁCIE A TVAROVEJ ANIZOTROPIE

V článku uvádzame výpočet Youngovho modulu pružnosti pre vzorky s periodicky rozloženými pórmi tvaru rotačných elipsoidov. Študujeme závislosť Youngovho modulu od celkovej pórovitosti, veľkosti, orientácie a od tvarovej anizotropie pórov. Ukazujeme, že prítomnosť pórov sploštených v smere zaťaženia významne znižuje hodnotu Youngovho modulu už pri nízkej pórovitosti.

Vypočítali sme taktiež Youngov modul ako funkciu pórovitosti pre systém, v ktorom sploštenosť pórov závisí od hodnoty celkovej pórovitosti. Teoretickú krivku porovnávame s experimentálnou krivkou, získanou pre Dy_2O_3.

1. Introduction

Most man-made construction materials contain pores. In order to ensure a fail-safe operation and reliability of machinery made from these materials, it is

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important to know the laws of the influence of porosity on mechanical properties of substances, particularly on moduli of elasticity.

The many-year development of theoretical mechanics of composite (consequently also porous) materials brought a wide range of valuable results in the form of theoretical methods of calculation of mechanical properties of such systems based on terms and mathematical apparatus of continuum mechanics [1–8].

On the other hand, specialists involved in practical studies of mechanical properties proposed a number of empirical and semiempirical relations between the modulus of elasticity and porosity [9–14]. The main reason of this was the relatively simple mathematical structure of the proposed relations (as compared with theoretical curves) that could fit to the experimental data.

Both above mentioned approaches use certain simplifying assumptions about the shape and distribution of pores. It is assumed that macroscopically quasi-homogeneous and quasi-isotropic specimens consisting of isotropic phases are involved in most cases.

The presented paper brings the results of theoretical calculations of dependence of Young's modulus not only on total porosity but also on the shape, size, and orientation of ellipsoidal pores which are periodically distributed throughout the sample. A significant decrease of Young's modulus is demonstrated already at low values of total porosity in the cases when the specimens contain pores that are flattened in the direction of the applied load. For one special case, where the correlation between total porosity and pore shapes is assumed, theoretical curve is compared with experimental data obtained from literature [15].

2. Calculation of Young's modulus for specimens with periodically arranged pores using methods of macroscopical theory of elasticity

The Young's modulus was calculated using the procedure analogical to that employed by Wang [11] and Phani and Niyogi [12]. Our calculations were carried out for a porous body of constant macroscopical cross-section A and length L. The application of tensile force W to this body in the direction of z axis results in the elongation ΔL. An effective Young's modulus E is then defined by the relation

$$E = \frac{\frac{W}{A}}{\frac{\Delta L}{L}}. \tag{1}$$

If $A(z)$ is the real cross-section of the body at the point z, i.e. the area of the section across "solid" parts excluding pores, then the elongation of the layer of a body $A(z)\mathrm{d}z$ is

$$\delta(z) = \frac{W}{A(z)E_0}\mathrm{d}z, \tag{2}$$

where E_0 is the Young's modulus of the material of the investigated body, i.e. the body without pores. The total elongation is then as follows:

$$\Delta L = \int_0^L \delta(z)\mathrm{d}z = \int_0^L \frac{W}{A(z)E_0}\mathrm{d}z. \tag{3}$$

By combining equations (1), (2) and (3) we obtain

$$\frac{E}{E_0} = \frac{L}{\displaystyle\int_0^L \frac{A}{A(z)}\mathrm{d}z}. \tag{4}$$

Similarly, porosity of a body is characterized by the relation

$$P = 1 - \frac{1}{AL}\int_0^L A(z)\mathrm{d}z. \tag{5}$$

In the process of deriving the equation (4) the transverse contraction of the body occurring at its longitudinal elongation is not taken into consideration. The lower is the longitudinal elongation, or Poisson's ratio, the higher is the accuracy of this approximation (omission). Theoretically, we can imagine infinitely small values of load W as well as of elongation ΔL, therefore expression (4) can be considered valid with sufficient accuracy.

In principle, relations (4) and (5) allow us to calculate the dependence of Young's modulus on porosity for real distribution and geometry of pores represented by the real cross-section of the body $A(z)$ as a function of the coordinate z. However, practical calculation can largely be carried out only for certain idealised types of geometry (e.g. periodical distribution of pores of simple geometrical shapes, a body produced by cubic arrangement of spherical polyhedrons [11], etc.).

In the real calculations, we used periodically distributed pores of ellipsoidal shape and all pores in the given specimen had the same size, shape and orientation; in that case $A(z)$ is a general periodic function.

For simplicity, we chose distribution of pores in the form of a simple cubic lattice with the "lattice" parameter a^*. In such a case the change in porosity is not caused by the change in the number of pores but by the change of their value at preservation of their number and shape (but not necessarily the maximum and minimum pore dimensions ratio – that depends on the problem studied). In order to simplify the calculations, we consider mostly isolated pores, however, in some cases also the pores interconnected in one direction. The details of the calculations are not presented. From the mathematical point of view some integrals are involved, sometimes very complex. We present only the final relations.

3. Pores in the form of spheroids

We consider pores in the shape of spheroids (Fig. 1). The half-axis in the direction of the rotational axis is b and the half-axis in the direction perpendicular to the rotational axis is a. The half-axis ratio is denoted $f = b/a > 0$. Some shapes of spheroid pores with different ratios f are in the Fig. 2. The angle between the rotational axis of pores and the direction of specimen tension is denoted as α. The exact direction of rotational axis of individual pores is unimportant because we have spheroids and because of the form of the equations (4) and (5). It is necessary to preserve the slope with regard to the axis of tension. Spheroids can therefore acquire different orientation in the specimens and the only requirement is that the slope of their rotational axis with regard to the direction of tension is α.

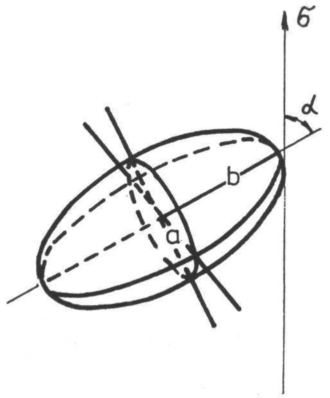

Fig. 1. A pore in the shape of a spheroid. Characteristic parameters: b – size of half-axis in the direction of rotational axis, a – size of half-axis in the direction vertical to the rotational axis, α – an angle between the rotational axis of an spheroid (pore) and the direction of specimen loading.

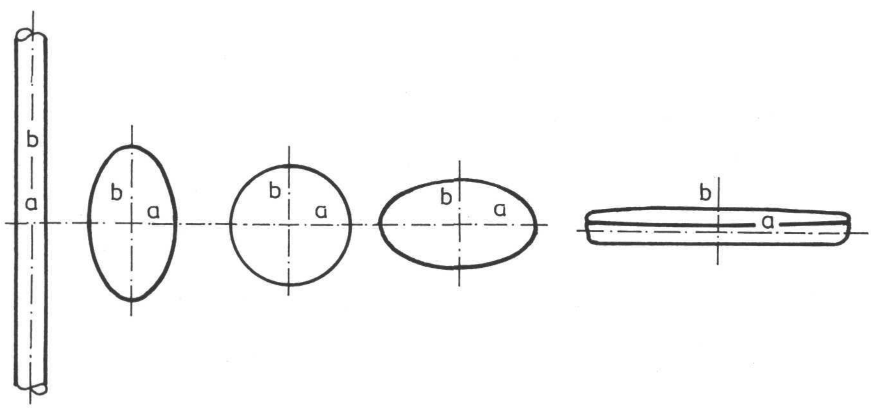

Fig. 2. Some shapes of spheroid pores with different ratios $f = b/a$. The f values for the examples illustrated (from left to right) are: $f \to \infty$, 2, 1, 0.5, 0.

In that case the normalized Young's modulus acquires the form

$$\frac{E}{E_0} = \frac{1}{1 - 2\xi + \frac{8}{3}\frac{\xi^2}{\sqrt{P}\sqrt{\frac{4}{3}\xi - P}} \arctan\sqrt{\frac{P}{\frac{4}{3}\xi - P}}}, \tag{6a}$$

where

$$\xi = \left(\frac{3}{4\pi}\frac{P}{f}\right)^{\frac{1}{3}} \sqrt{\sin^2\alpha + f^2\cos^2\alpha},$$

while parameters P, f and α must satisfy the following conditions:

$$\frac{P}{f}(\sin^2\alpha + f^2\cos^2\alpha)^{\frac{3}{2}} \leq \frac{\pi}{6}$$

and at the same time

$$\frac{P}{f}(f^2\sin^2\alpha + \cos^2\alpha)^{\frac{3}{2}} \leq \frac{\pi}{6}.$$

If the parameters comply with the specified conditions closed isolated pores are involved.

Using A_c (A_c is a ratio of the minimum load-bearing cross-section of the sample, i.e. the minimum value of $A(z)$, to the geometrical cross-section A; so $0 \leq A_c \leq 1$), the expression is as follows:

$$\frac{E}{E_0} = \frac{1}{1 - 2\frac{(\sin^2\alpha + f^2\cos^2\alpha)^{\frac{3}{4}}}{(\pi f)^{\frac{1}{2}}}\left[\sqrt{1 - A_c} - \frac{1}{\sqrt{A_c}}\arctan\sqrt{\frac{1 - A_c}{A_c}}\right]}, \tag{6b}$$

while parameters A_c, f, α must satisfy the following conditions:

$$\frac{1 - A_c}{f}(\sin^2\alpha + f^2\cos^2\alpha)^{\frac{3}{2}} \leq \frac{\pi}{4}$$

and at the same time

$$\frac{1 - A_c}{f}(\sin^2\alpha + f^2\cos^2\alpha)^{\frac{1}{2}}(f^2\sin^2\alpha + \cos^2\alpha) \leq \frac{\pi}{4}.$$

The quantity A_c can serve as a qualitative estimation of the mean distance between pores in the sample. The higher A_c corresponds to the higher mean distance between pores.

If $f = 1$ we obtain a special case – spherical pores. Then the Young's modulus obeys the following expression:

$$\frac{E}{E_0} = \frac{1}{1 - \left(\frac{6}{\pi}P\right)^{\frac{1}{3}} + \frac{2}{\sqrt{\pi}\sqrt{1-\left(\frac{3}{4}\sqrt{\pi}P\right)^{\frac{2}{3}}}} \arctan \frac{\left(\frac{3}{4}\sqrt{\pi}P\right)^{\frac{1}{3}}}{\sqrt{1-\left(\frac{3}{4}\sqrt{\pi}P\right)^{\frac{2}{3}}}}}, \quad P \le \frac{\pi}{6}. \tag{7a}$$

Expressed in terms of A_c

$$\frac{E}{E_0} = \frac{1}{1 - \frac{2}{\sqrt{\pi}}\left(\sqrt{1-A_c} - \frac{1}{\sqrt{A_c}} \arctan \sqrt{\frac{1-A_c}{A_c}}\right)}, \quad 1 - A_c \le \frac{\pi}{4}. \tag{7b}$$

We derive further an expression for the Young's modulus of a body with pores interconnected in the direction of tension, however, not in the direction perpendicular to the tension in the case when the rotational axis of spheroids is oriented in the direction of tension ($\alpha = 0$). The expression is as follows:

$$\frac{E}{E_0} = \frac{\sqrt{\pi}\sqrt{12f^2(1-P)-\pi}}{4\sqrt{3}f^2 \arctan \sqrt{\frac{3\pi}{12f^2(1-P)-\pi}}} \tag{8a}$$

and the relevant parameters have to satisfy conditions $f \ge 1$ and

$$\frac{\pi}{6} \le f^2 P \le \pi\left(\frac{f^2}{4} - \frac{1}{12}\right).$$

Expressed in terms of A_c

$$\frac{E}{E_0} = \frac{\sqrt{\pi}\sqrt{A_c}}{2f \arctan \sqrt{\frac{\pi}{4f^2 A_c}}}, \quad \frac{\pi}{4}\frac{1}{f^2} \le 1 - A_c \le \frac{\pi}{4}. \tag{8b}$$

4. Comparison with experimental data

Verification of correctness of our results is rather complicated because, besides the total porosity data, we also need the data about the shape or, as the case may be, about orientation of pores.

Results concerning the dependence of Young's modulus on the flattening of pores (illustrated for various situations in Figs. 3, 4, 5) are confirmed (at least qualitatively) by the explanation of Dean and Lopes [16] of the origin of positive curvature of the Young's modulus-porosity curve, observed in ceramic materials. In opinion of Dean and Lopes the pores in ceramic materials show a tendency to become more spherical with the increasing porosity. This increase in the spherical character of pores with the increase of total porosity was also observed experimentally in microphotographs of thin sections of MgO [17].

Theoretical studies carried out in low-porosity materials revealed that the slope of the Young's modulus-porosity curve increases with the flattening of pores (the same result, from the qualitative point of view, was obtained in this article

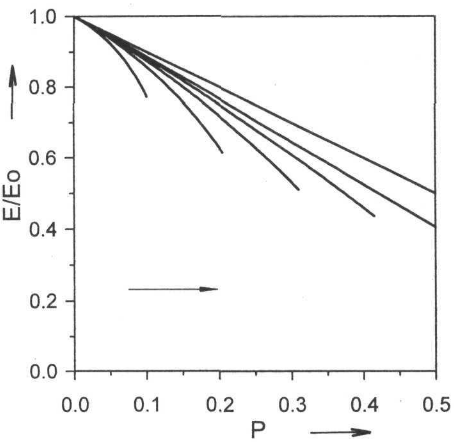

Fig. 3. Dependence of a normalized Young's modulus E/E_0 on the porosity P for a model system with isolated pores in the shape of spheroids with rotational axis oriented in the direction of loading and varying values of the ratio f between dimensions of pores in the direction of the rotational axis and perpendicular to the axis. For the curves shown in the picture the value of f increases in the direction of the arrow and acquires values 0.2, 0.4, 0.6, 0.8, 1.0, and $f \rightarrow \infty$.

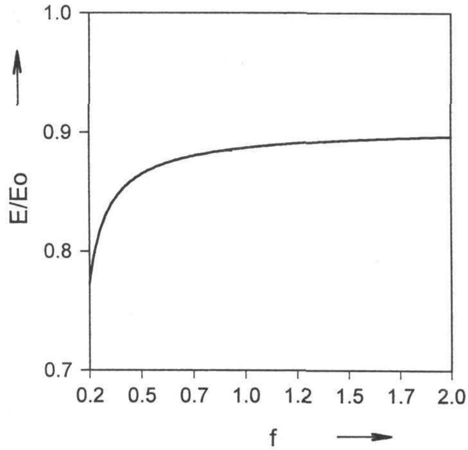

Fig. 4. Normalized Young's modulus E/E_0 of a model system with pores in the shape of spheroids with rotational axis oriented in the direction of loading as a function of the ratio f between the dimension of an spheroid in the direction of the rotational axis and the direction perpendicular to the axis for porosity $P = 0.1$.

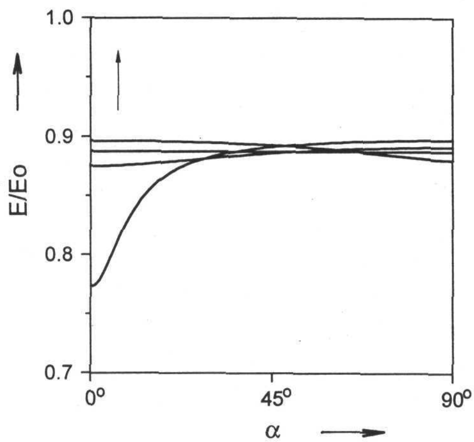

Fig. 5. Normalized Young's modulus E/E_0 of a model system with pores in the shape of spheroids as a function of an angle between the loading axis and rotational axis of pores for various values of the ratio f between the dimensions of pores in the direction of their rotational axis and the direction perpendicular to the axis at total porosity $P = 0.1$. For the curves shown in the picture the value of f increases in the direction of arrow and acquires values 0.2, 0.6, 1, 2.

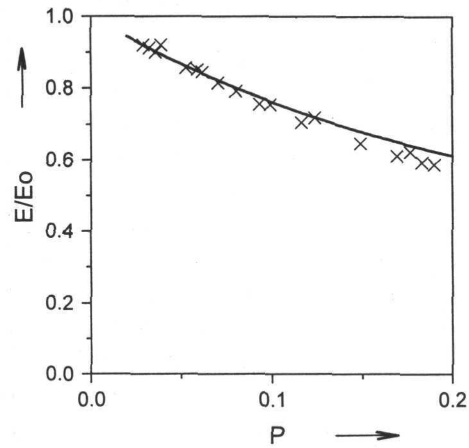

Fig. 6. Normalized Young's modulus E/E_0 as a function of porosity P. Comparison of experimental data for Dy_2O_3 (crosses) with theoretical data (solid line) calculated (according to 6a) for the system with pores in the shape of flattened spheroids. The flattening decreases linearly with growing porosity ($f = 0.04$ for $P = 0.02$; $f = 0.382$ for $P = 0.2$).

for periodically distributed pores). Thus, if the flattening of pores decreases with the growth of porosity, the slope of the Young's modulus-porosity plot decreases, too, and the resulting curve exhibits a positive curvature. This idea is illustrated in Fig. 6. In this picture there are plotted the experimental data for the Young's modulus of Dy_2O_3 and the theoretical curve obtained according to our equation (6a) for the Young's modulus of the specimen with pores in the shape of flattened spheroid the flattening of which decreases with the increase of total porosity. We used a linear growth of the ratio of the shorter and longer axis of a spheroid, ranging from 0.04 at two per cent porosity to approx. 0.38 at twenty per cent porosity, as the simplest model in such a case. Both behaviours show a qualitative agreement.

5. Discussion and conclusions

The presented study was intended as a theoretical calculation of Young's modulus of elasticity of porous materials as a function of porosity (if necessary, also of additional characteristics of pores – shape, orientation) for a specimen with periodically arranged pores.

In our calculations we used an equation expressing Young's modulus of elasticity as a functional of the real area of the specimen cross-section, i.e. the area of cross-section intersecting the "solid parts" and not the pores as published in [11, 12]. In this approach, the sample with any (low or high) pore concentration is treated as a whole. All pores (isolated or interconnected) contained in the given sample with their "interactions", real geometrical shapes, and spatial distributions are taken into account.

To perform analytical calculations, a system consisting of a continuous matrix containing periodically arranged identical pores was used as a model system. Changes in porosity were attained by changing the size of pores while their number, shape and orientation remained constant.

The expressions for the Young's modulus as a function of total porosity (or other more detailed characteristics of pores such as the ratio of their dimensions in the direction of tension and perpendicular to tension, orientation, etc.) were derived for ellipsoidal pores either isolated or interconnected.

The system with pores of an anisotropic shape (i.e. pores with rather different maximum and minimum dimensions) exhibits, for the given value of total porosity, rather high sensitivity of Young's modulus to orientation of pores with regard to the direction of the applied tension. The value of Young's modulus is higher for the pores "elongated" in the direction of tension than for the pores "flattened" in that direction. This is illustrated by the calculation of the Young's modulus for both the system with pores changing gradually from "flattened" spheroids to elongated ones while the rotational axis remains in the direction of tension (Fig. 3, 4), and the system with pores in the shape of flattened spheroids, however, with varying slope of rotational axis with regard to the direction of tension (gradual change from the flattened to the elongated pores) (Fig. 5).

The verification of applicability of our results (for periodically arranged pores) is more complex because, besides the data on total porosity, we need to know the character of porosity (shape of pores, anisotropy of the shape, orientation of pores, etc.) for the real specimen. The explanation of the origin of positive curvature of the experimentally measured Young's modulus-porosity curves in the ceramic materials presented in the study [16] speaks in support of our results. The authors of the study mentioned are of the opinion that the reason for positive curvature is the increase in sphericity of pores with the growth of total porosity. Fig. 6 shows a qualitative agreement between experimental data and the curve obtained on the basis of our results for the Young's modulus of a specimen with pores in the shape of flattened spheroids provided that the ratio of the length of the shorter and longer axis of an ellipsoid increases linearly with the porosity.

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REFERENCES

[1] HASHIN, Z.—SHTRIKMAN, S.: J. Mech. Phys. Solids, *10,* 1962, p. 335.
[2] HASHIN, Z.—SHTRIKMAN, S.: J. Mech. Phys. Solids, *11*, 1963, p. 127.
[3] CHRISTENSEN, R. M.: J. Mech. Phys. Solids, *38*, 1990, p. 379.
[4] ZIMMERMAN, R. W.: Mech. Mat., *12*, 1991, p. 17.
[5] BUDIANSKY, B.: J. Mech. Phys. Solids, *13*, 1965, p. 223.
[6] BERRYMAN, G.: J. Energy Resources Techn., *116*, 1994, p. 87.
[7] HILL, R.: J. Mech. Phys. Solids, *13*, 1965, p. 213.
[8] MORI, T.—TANAKA, K.: Acta Metallurgica, *21*, 1973, p. 571.
[9] SPRIGGS, R. M.: J. Am. Cer. Soc., *44*, 1961, p. 628.
[10] HASSELMAN, D. P. H.: J. Am. Cer. Soc., *45,* 1962, p. 452.
[11] WANG, J. C.: J. Mat. Sci., *19*, 1984, p. 801.
[12] PHANI, K. K.—NIYOGI, S. K.: J. Mat. Sci., *22*, 1987, p. 257.
[13] BOCCACCINI, A. R.—ONDRACEK, G.—MAZILU, P.—WINDELBERG, D.: J. Mech. Behavior Mater., *4*, 1993, p. 119.
[14] KUPKOVA, M.: J. Mat. Sci., *28*, 1993, p. 5265.
[15] RAMAKRISHNAN, N.—ARUNACHALAM, V. S.: J. Am. Ceram. Soc., *76*, 1993, p. 2745.
[16] DEAN, E. A.—LOPEZ, J. A.: J. Am. Ceram. Soc., *66*, 1983, p. 366.
[17] JANOWSKI, K. R.—ROSSI, R. C.: J. Am. Ceram. Soc., *50*, 1967, p. 599.

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